# Newton's Law of Cooling

## 1 Exponential Growth and Decay

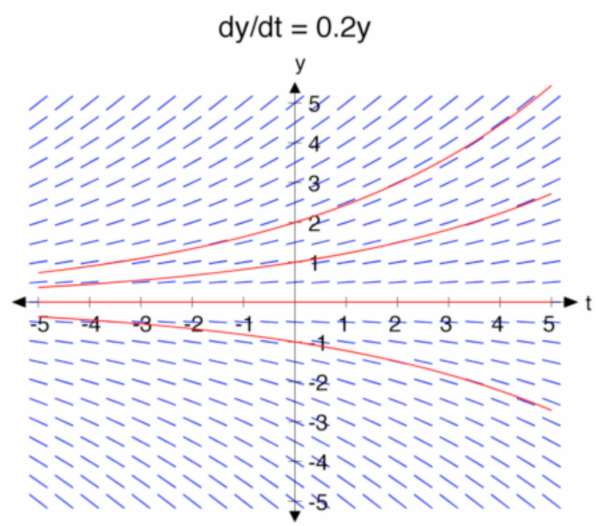
Recall the differential equation that we used to model exponential growth and decay.

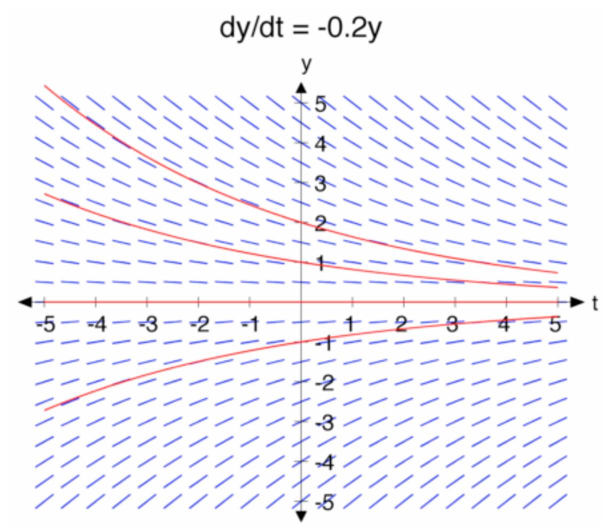
$$\frac{dy}{dt} = ky$$

Solutions are $y(t) = c \cdot e^{kt}$ for any constant $c$.

If $k > 0$, this is exponential growth.

If $k < 0$, this is exponential decay.




## 2 Statement of the Law

Newton's Law of Cooling states that

**The rate of change of the temperature of a warming/cooling body is proportional to the difference between the temperature of the body and the surrounding temperature.**

If we let $y(t)$ denote the temperature of the body at time $t$ and $a$ denote the temperature of the surroundings, we arrive at the following differential equation:

$$\frac{dy}{dt} = -k(y - a)$$

Observe that if $k > 0$, we have the following (sensible) relationships between $\frac{dy}{dt}$ and $y - a$:

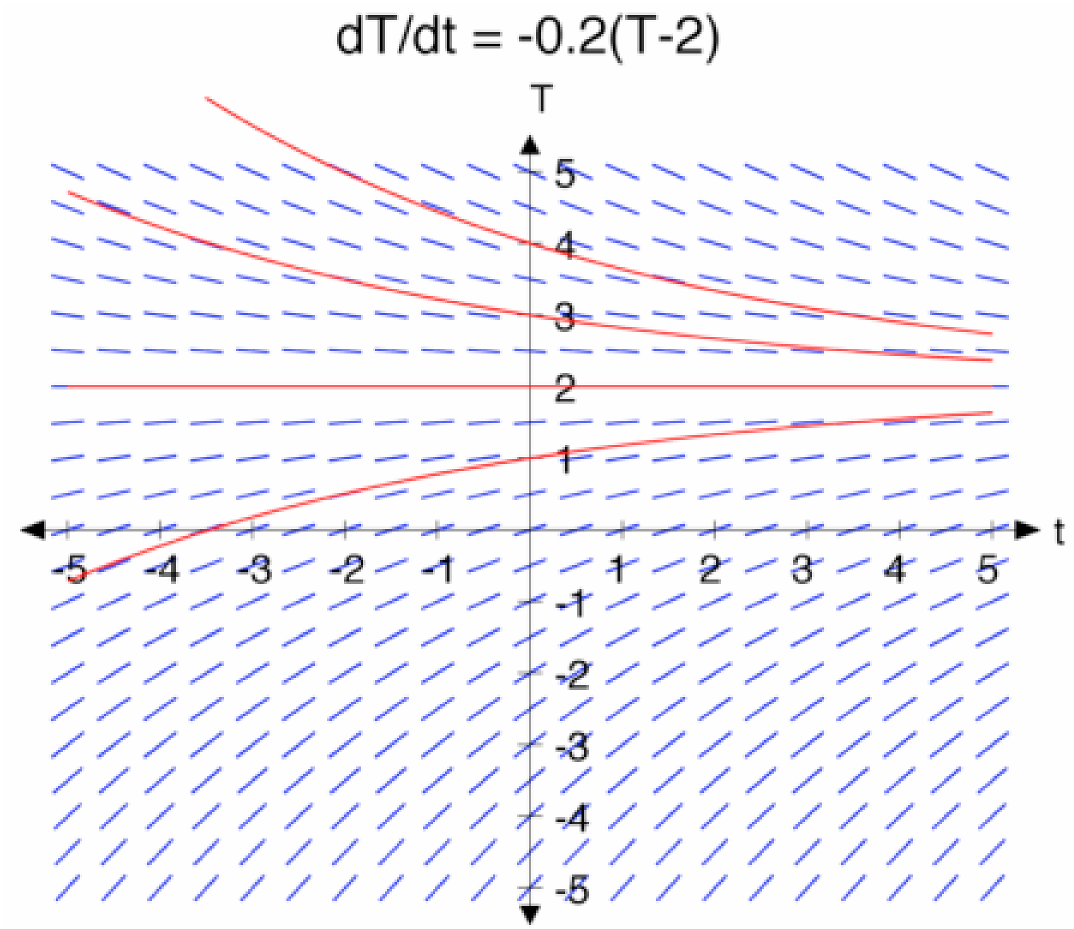
- if the body is hotter than the surroundings ($y - a > 0$), then the temperature of the body will be decreasing ($\frac{dy}{dt} < 0$)
- if the body is cooler than the surroundings ($y - a < 0$), then the temperature of the body will be increasing ($\frac{dy}{dt} > 0$)

## 3 Example

If we let $k = 0.2$ and $a = 2$, we have the following differential equation.

$$\frac{dy}{dt} = -0.2(y - 2)$$

By computing the slope field for this differential equation and sketching some solutions, it seems plausible that the solutions are shifted (up by the surrounding temperature) versions of the exponential decay functions in 1.




It is a straightforward computation to check that $y(t) = 2 + c \cdot e^{-0.2t}$ is a solution to $\frac{dy}{dt} = -0.2(y - 2)$ for any choice of $C$.

In general, for any body whose temperature is governed by Newton's Law of Cooling:

$$\frac{dy}{dt} = -k(y - a)$$

The solutions to this equation are given by the shifted exponential decay functions:

$$y(t) = a + c \cdot e^{-kt}$$

If we are given an initial condition, $y(0) = y_0$, we can solve for $c$:

$$y(0) = a + c \cdot e^{-k(0)} = y_0 \qquad \text{so} \qquad c = y_0 - a$$

Thus the (unique) solution to the initial value problem

$$\frac{dy}{dt} = -k(y-a)$$
$$y(0) = y_0$$

is given in terms of the constant $k$, the surrounding temperature $a$, and the initial temperature $y_0$ by the equation:

$$y(t) = a + (y_0 - a)e^{-kt}$$

## 4 Exercises

1. A cup of coffee is made with boiling water at a temperature of 100 °C, in a room at temperature 20 °C. After two minutes it has cooled to 80 °C. What is its temperature after four minutes? When will the coffee drop below 40 °C and taste cold?

   (Solution):

   Let $y(t)$ measure the temperature (in °C) of the cup of coffee at time $t$ (in minutes after it was poured). The differential equation and initial condition that $y(t)$ satisfies is:

   $$\frac{dy}{dt} = -k(y-20) \qquad \text{(differential equation)}$$
   $$y(0) = 100 \qquad \text{(initial conditon)}$$

   Based on the differential equation, we know that the solutions are shifted exponential decay functions.

   $$y(t) = 20 + c \cdot e^{-kt}$$

   Now we only need to solve for $c$ and $k$.

   From the initial condition, we can solve for $c$:

   $$y(0) = 20 + c \cdot e^{-k(0)} = 100 \qquad \text{so} \qquad c = 80$$

   Now we know that the solution looks like:

   $$y(t) = 20 + 80e^{-kt}$$

   We can use the second piece of information ($y(2) = 80$) to solve for $k$:

   $$y(2) = 20 + 80e^{-k(2)} = 80 \qquad \text{so} \qquad k = \frac{-\ln(60/80)}{2} \approx 0.1438$$

   Thus the formula for $y(t)$ is:

   $$\boxed{y(t) = 20 + 80e^{-0.1438t}}$$

   After 4 minutes, the coffee will be $y(4) = 20 + 80e^{-0.1438(4)} \approx 65.007°C$

2. You take an ice-cream out of the freezer, kept at $-4°C$. Outside it is $32°C$. After one minute, the ice-cream has warmed to 8°C. What is the temperature of the ice-cream after five minutes?

3. A delicious pumpkin pie is removed from an oven that is set at $162°C$ and is placed on a rack to cool. The temperature in the kitchen is $21°C$. Letting $T(t)$ represent the temperature of the pie at any time $t$ (in minutes), Newton's Law of Cooling says that the rate at which the pie cools is described as

$$\frac{dT}{dt} = -k(T - 21), \qquad \text{where } T(0) = 162.$$

   *a.*) Show that the function $T(t) = 21 + 141e^{-kt}$ satisfies the differential equation and the initial value.

   *b.*) Suppose the temperature of that tasty pie is $91°C$ after 30 minutes of resting on the cool rack. Use part (a) and the new information provided to find the specific formula for the temperature of the pie at time $t$.

4. A detective is called to the scene of a crime where a dead body has just been found. She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be $80°F$. The detective checks the programmable thermostat and finds that the room has been kept at a constant $68°F$ for the past 3 days.

   After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be $78°F$. This last temperature reading was taken exactly one hour after the first one. The next day the detective is asked by another investigator, "What time did our victim die?". Assuming that the victim's body temperature was normal ($98.6°F$) prior to death, what is her answer to this question?

5. Suppose the solution to a cooling body problem is found to be

$$T(t) = 21 + 9e^{-\frac{1}{4}t},$$

   *a.*) Determine the differential equation that this function satisfies and show that it does indeed satisfy the equation.

   *b.*) What is the initial temperature of the body? What is the temperature of the surrounding environment?

   *c.*) What happens to the temperature as $t \to \infty$?

   *d.*) If $t$ is in minutes, what is the temperature of the body after 20 minutes?

   *e.*) What is the rate of change of the temperature after 20 minutes?

6. Find the solution to each of the following initial value problems:

   *a.*) $dy/dt = -7y$, with $y = 20$ at $t = 0$.

   *b.*) $dy/dt = -7(y - 10)$, with $y = 20$ at $t = 0$.

   *c.*) $dy/dt = -7y + 14$, with $y = 20$ at $t = 0$.

   *d.*) For each solution above, verify that it is a solution by plugging it in the differential equation and making sure both sides are indeed equal.