The Definition of e

1 Introduction

e is a fundamental contstant in mathematics, named after the renowned and prolific mathematician Leonhard Euler (1707-1783). It is often first encountered when taking a limit as $n \to \infty$ of the quantity $(1 + \frac{r}{n})^n$, which approaches e^r and describes the growth of money in a bank account when the interest is "continuously compounded" at interest rate r.

The above example motivates important applications to exponential growth and decay models, and hints at the essential role played by e in calculus (especially differential equations). In this note, we will understand how the number e is defined to be the base of the unique exponential function which is its own derivative. For this reason e is also known as the base of *the* exponential function, e^x.

In Math 132 Calculus 2, you can see an altogether different definition of e in terms of an integral, which is why e is also the base of the *natural* logarithm function $\ln(x) = \log_e(x)$. In case you are curious, the term "Early Transcendentals" (which is often on the covers of calculus texts) refers to introducing e before its "correct" place in Calc 2.

2 Exponential functions

We say that a function f is an **exponential function** if $f(x) = b^x$ for some positive real number b, the "base".

In the special case when $b = 1$, then $f(x) = 1$ is just a constant function, and is not considered to be exponential.

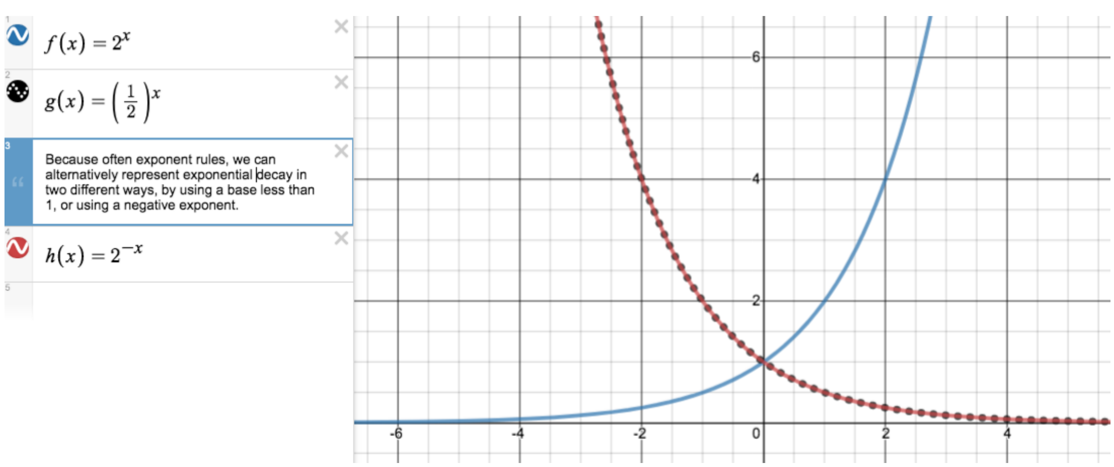
If the base b is larger than 1, then $f(x) = b^x$ is always increasing, and we say f represents **exponential growth**.

If the base is between 0 and 1, then $f(x) = b^x$ is always decreasing, and we say f represents **exponential decay**. Note that for exponential decay, $f(x)$ may also be written as $f(x) = B^{-x}$, in terms of a base B which is greater than 1, by letting $B = \frac{1}{b} = b^{-1}$.

Regardless of the base, an important property of exponential functions is that they are always positive, so $b^x > 0$ for every x, and that the values of the limits as $x \to \infty$ and $x \to -\infty$ will be ∞ and 0 (or 0 and ∞, when the base is less than 1). It is also important that for every $b \neq 1$, the $f(x)$ passes the horizontal line test, and is thus one-to-one, and invertible. We define the logarithm base b function to be the inverse function to $f(x)$, so if $f(x) = b^x$, then $f^{-1}(x) = \log_b(x)$.

Notice that if you write out the essential property of inverses $\Big(y = f(x)$ if and only if $x = f^{-1}(y) \Big)$ in the context of these exponential and log functions, you will recover the "log base answer equals exponent" mnemonic for remembering how logs work.

As an example, $f(x) = 2^x$, and $g(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$ are graphed below, and are prototypical examples of the graphs of exponential growth (in blue) and exponential decay (in red/black). It is important to notice that although $g(x) = 1/f(x) = (f(x))^{-1}$, f and g are not each others inverse, but their formulas are reciprocals.



3 Derivatives of exponential functions

Suppose $b > 0$ is any base. In order to determine the derivative of the exponential function $f(x) = b^x$, we use the definition:

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \;=\; \lim_{h\to 0} \frac{b^{x+h}-b^x}{h}$$

Using properties of exponents and factoring, we can simplify the expression $\frac{b^{x+h}-b^x}{h} = \frac{b^x \cdot b^h - b^x}{h} = \frac{b^x\left(b^h-1\right)}{h} = b^x \cdot \frac{b^h-1}{h}$

Since b^x is a constant with respect to h, we can use limit rules to factor it out of the limit

$$f'(x) \;=\; \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} \;=\; \lim_{h\to 0}\left(b^x \cdot \frac{b^h-1}{h}\right) \;=\; b^x \cdot \lim_{h\to 0} \frac{b^h-1}{h}$$

Now we can see that if the limit $\lim_{h\to 0} \frac{b^h-1}{h}$ exists, say it is equal to the number L, then $f'(x) = L \cdot f(x)$. In other words, the derivative of the exponential function is just a constant multiple of the function itself.

It turns out that this limit *does* exist, and is equal to the number which is called "the natural log of b" and denoted $\ln(b)$ or $\log_e(b)$, but you will have to take this as a fact until you get to Calc 2 and can more fully understand the connections between these functions.

Thus, we have for every exponential function $f(x) = b^x$, that the derivative is given by $f'(x) = \ln(b) \cdot b^x$.

For example, when $b = 2$, the derivative of $f(x) = 2^x$ is $f'(x) = \ln(2) \cdot 2^x \approx 0.693 \cdot 2^x$ since $\ln(2) = 0.69314718056\ldots$

If $b = \frac{1}{2}$, then the derivative of $g(x) = \left(\frac{1}{2}\right)^x$ is $g'(x) = \ln(\frac{1}{2}) \cdot \left(\frac{1}{2}\right)^x \approx -0.693 \cdot \left(\frac{1}{2}\right)^x$. Note that since we can also write $g(x) = 2^{-x}$, which is the composition of 2^x and $-x$, we can use the Chain Rule to see that $g'(x) = \ln(2) \cdot 2^{-x} \cdot (-1)$, which is the same as the first expression for $g'(x)$, since $-\ln(2) = \ln(2^{-1})$.

4 *The* exponential function

By taking for granted that the final limit in the definition of the derivative for b^x works out to be $\ln(b)$, and also taking as fact that the function $\ln(x) = \log_e(x)$ is the inverse function to $f(x) = e^x$, it is easy to see that the function $f(x) = e^x$ has the special property of being its own derivative $f' = f$, since from above we know that $f'(x) = \ln(e) \cdot e^x = 1 \cdot e^x = e^x$ and $\ln(e) = f^{-1}\big(f(1)\big) = 1$.

Although it is beyond our current abilities to understand why the limit $\lim_{h\to 0} \frac{b^h-1}{h}$ works out to $\ln(b)$, we can easily make tables of values for different choices of b and h in order to convince ourselves that the limit exists, and that there should be some special base which makes the limit equal to 1.

The following table can be filled in based on some different choices of b with several choices of h close to 0 for each. The true value of $\ln(b)$ is given in the center for comparison.

$\frac{b^h-1}{h}$	$h = -0.1$	$h = -0.01$	$h = -0.001$	$\ln(b)$	$h = 0.001$	$h = 0.01$	$h = 0.1$
$b = 1.5$				$0.40546510\ldots$			
$b = 2$				$0.69314718\ldots$			
$b = 2.5$				$0.91629073\ldots$			
$b = 3$				$1.09861228\ldots$			
$b = 3.5$				$1.25276296\ldots$			
$b = 4$				$1.38629436\ldots$			

Because it is so easy to take derivatives of e^x, it is often convenient to express other exponential functions in terms of $f(x) = e^x$. Since $e^{\ln(b)} = b$ for any b, we can always write $g(x) = b^x$ as $g(x) = \left(e^{\ln(b)}\right)^x = e^{\ln(b)\cdot x} = f\big(\ln(b) \cdot x\big)$. Thus, all exponential functions can be represented as horizontally scaled (and horizontally reflected in the case of decay) transformations of $f(x) = e^x$.