Introduction to Differential Equations,
Initial Value Problems, and Exponential Growth Models

1 Differential Equations (DE's) and Initial Value Problems (IVP's)

In "Video TT2a: Differential Equations" on the Trinity Math Department YouTube page, we introduced that a **differential equation** is an equation that relates an unknown function (usually denoted y) with it's derivatives. A **solution** to a differential equation is a *function* which makes the equation true.

Also in the video, we introduced that an **initial value problem** is a differential equation, together with an **initial condition**. An initial condition is a constraint of the form $y(x_0) = y_0$, which specifies a point (x_0, y_0) that the unknown function must pass through. A **solution** to an initial value problem is a *function* which makes both the differential equation and the initial condition true.

Although the situation is more complicated in general (see Math 234 - Differential Equations), in this class all of the differential equations we will see have an infinite number of solutions, and all of the initial value problems we will see have exactly one solution.

In this class, we will not worry about how to solve differential equations (except for a few very special cases), but will focus on the following:

- What it means for a function to be a solution to a DE (or IVP), and how to check whether a given function satisfies a given DE (or IVP). This is exactly what is covered in the YouTube **Video TT2a: Differential Equations**.
- How, in certain important cases, a differential equation can (should) be interpreted as a sentence in the English language which conveys useful information about the relationship between the unknown function and its derivative.
- How the statement that "exponential functions are always proportional to (a constant multiple of) their derivatives", is equivalent to the central idea underlying exponential growth and decay models.

2 Exponential Growth Models

There are many situations in the real world where numerical quantities are well modeled by exponential functions. We will investigate some specific examples below, and in the subsequent Interludes, but for now we will just recall the main ideas.

Before talking about exponential growth, it is useful to first think about the simpler case of **linear growth**. The underlying idea of linear growth is that the growth happens at a constant rate.

2.1 Linear Growth

If $f(t) = mt + b$ represents how some quantity $f(t)$ is a function of time t, then we say that the quantity is linear (as a function of time).

Some important observations:

- $f(0) = m(0) + b = b$, so the "initial" value (at time $t = 0$) is the same as the constant b.
- $f'(t) = m$, so the derivative is always equal to the constant m, and the quantity is always changing at the same rate.
- The two items above give us a way to characterize lines in terms of differential equations. A function is a line if and only if it is a solution to an initial value problem of the form $\left(\ y' = m,\ y(0) = b\ \right)$, for two constants m and b. In other words, if it starts at b (when $t = 0$), and always grows at a constant rate, where $y'(t) = m$ for all t.
- Given any two points in the xy-plane (with different x-coordinates), there is exactly one function of the form $y = mx + b$ which passes through these points. It is an exercise in high school alegbra to determine how the constants m and b depend on the coordinates of the points.

2.2 Exponential Growth

If $f(t) = Ae^{kt}$ represents how some quantity $f(t)$ is a function of time t, then we say that the quantity exhibits **exponential growth** (or **exponential decay** in the case that k is negative).

Although the term "exponential growth" is used in everyday language whenever the rate of growth is ever increasing, the precise definition is that the rate of growth of a quantity is always proportional to the quantity itself (and the constant of proportionality is fixed).

Some important observations:

- $f(0) = Ae^{k(0)} = Ae^0 = A$, so the "initial" value (at time $t = 0$) is the same as the constant A.
- $f'(t) = A \cdot ke^{kt} = k \cdot f(t)$, so the derivative is proportional to the function. Moreover, the derivative at any time is always exactly k times the value of the function at that time.
- The two items above give us a way to characterize exponential functions in terms of differential equations. A function is an exponential function if and only if it is a solution to an initial value problem of the form $\big(\ y' = ky,\ y(0) = A\ \big)$, for two constants k and A. In other words, if it starts at A (when $t = 0$), and grows in such a way that $y'(t) = ky(t)$ for all t.
- Given any two points in the xy-plane (with different x-coordinates), there is exactly one function of the form $y = Ae^{kx}$ which passes through these points. In one of the exercises below, you will be asked to solve for A and k in terms of the coordinates of the two points.

3 Exercises

1. Determine whether each of the following statements is **True** of **False**. Explain how you know in either case.
   (a) $y(x) = 2e^{2x}$ is a solution of the differential equation $\frac{dy}{dx} = 4y$.
   (b) $R(t) = (2t+1)^2$ is a solution of the differential equation $\frac{dR}{dt} = \frac{2R}{t+1}$.
   (c) $g(t) = 5e^{-3t}$ is the solution of the initial value problem $g' = -3g, \quad g(0) = -3$.
   (d) $y(t) = -3e^{-3t}$ is the solution of the initial value problem $y' = -3y, \quad y(0) = -3$.
2. At $t = 0$, a certain abandoned nuclear test site was quarantined, when it was observed that 417.870 kilograms of radioactive material was left behind. 10 minutes later, a second measurement detected 417.855 kilograms.
   (a) Given the fact that the amount of radioactive material decays exponentially as a function of time, find an exact formula for $R(t)$, the number of kilograms of material, t minutes after the initial time $t = 0$.
   (b) Using your formula from (a), determine the "half-life" of the radioactive material (the amount of time it takes for the mass to decrease by half).
   (c) It will not be safe to return to the site until there is 10 kilograms or less. How many years will it take until it is safe to visit the site?
3. The value $V(t)$ of a certain model of car which costs \$60,000 is known to depreciate at a rate of 20% per year, where t is age of the car in years.
   (a) Based on the information given in the problem, calculate $V(0)$, $V(1)$ and $V(2)$ and interpret these numbers in the context of the problem. Write down a formula for $V(t)$.
   (b) Using your formula from (a), determine a formula for $V'(t)$, and use it to evaluate $V'(0)$ and $V'(1)$. Interpret these numbers in the context of the problem.
   (c) Use your formula from (a) to show that no matter what time you start watching the value of the car (say at some time $t = c$), after one year has passed (at time $t = c + 1$) the value will have decreased by 20%.
   (d) Write down a differential equation that describes the evolution of V. Write down the initial condition for this problem.

4. The number of bacteria in a certain population was observed this morning at 8:04am to be 10.23 million. The population has been observed to doubling every 6 minutes. Let $P(t)$ represent the number of millions of bacteria in the population, where t is the number of minutes since this mornings observation was made.

   (a) Based on the information given in the problem, calculate $P(0)$, $P(1)$ and $P(2)$ and interpret these numbers in the context of the problem. Assuming the growth continues to double every 6 minutes, Write down a formula for $P(t)$.

   (b) Assuming that this entire population started from a single bacterium, at what time did that first bacterium start multiplying.

   (c) Use your formula from (a) to show that no matter what time you start watching the population, after 6 minutes have passed the population will have doubled.

   (d) Write down a differential equation that describes the evolution of P. Write down the initial condition for this problem.

5. The efficiency of solar panels as a function of time has been studied and roughly degrades at a rate of 1% efficiency per year (for any one year interval, the panels are 1% less efficient at the end of the year than at the beginning of the year). Let $E(t)$ measure the efficiency (as a percentage of the original efficiency, out of 100 percent) as a function of t, the number of years that have passed since the installation.

   (a) Based on the information given in the problem, calculate $E(0)$, $E(1)$ and $E(2)$ and interpret these numbers in the context of the problem. Write down a formula for $E(t)$.

   (b) Use your formula from (a) to show that no matter what time you start watching the panels, after one year has passed the efficiency will have decreased by 1%.

   (c) Write down a differential equation that describes the evolution of E. Write down the initial condition for this problem.