Applications of Exponential Growth Models and Differential Equations to Finance

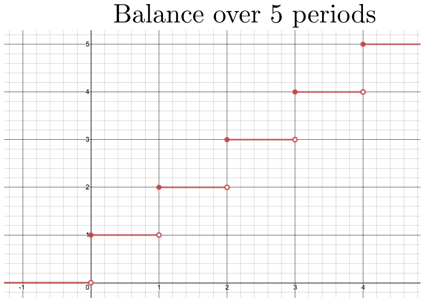
The first application we will consider is the evolution of the amount of money $M(t)$ (in dollars) in a bank account, as a function of time. We will consider the effects of making regular deposits to the account, and then the effect of compounding interest at a fixed interest rate r. Usually, these actions happen at regular intervals (once a month, or once a day, week, year, etc. depending on the type of account), but it is very useful to sometimes assume that these actions happen "continuously", which we will investigate below.

1 Making deposits to a bank account

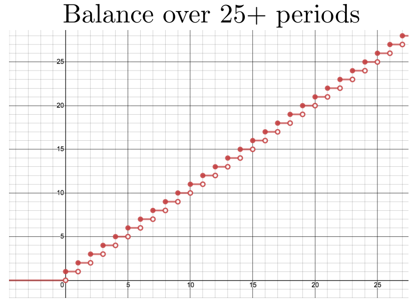
Imagine that you open a bank account with an initial deposit of \$1. If you deposit one addition dollar each period (day), your balance $M(t)$ at time t is a piecewise constant function. In fact it is exactly equal to the "floor" of $t+1$, sometimes denoted floor$(t+1)$, and is the number obtained when $t+1$ is "rounded down" to the nearest whole number.

The first set of three figures below show that when you are looking at large enough time scales, it makes sense (and is very convenient) to think of the money in the bank account as growing linearly (continuously), even though the balance is actually a piecewise constant function, with right-continuous jumps every time you make a deposit.

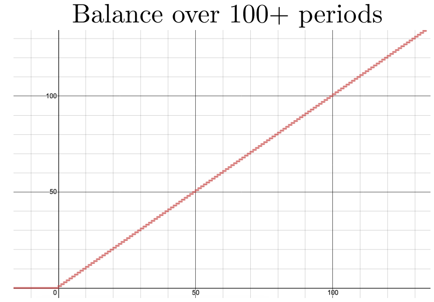
Balance over 5 periods



Balance over 25+ periods

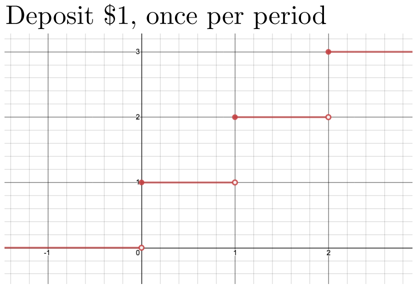


Balance over 100+ periods

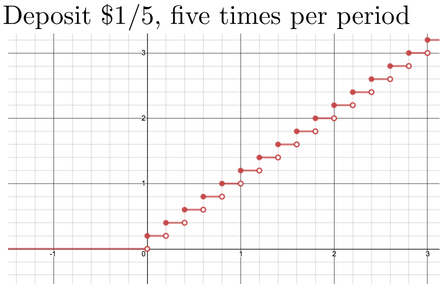


The three figures below show that even if you are looking at a fixed number of time periods, if we split each deposit into m equal parts, and also split each period into m equal parts, we can deposit $1/m$ dollars at each of the m sub-periods, and end up depositing 1 dollar by the end of the period, as before. As before it makes sense to approximate the behavior with a linear function.

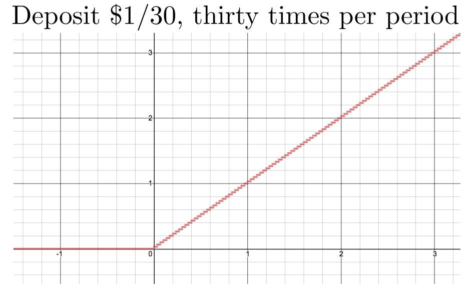
Deposit \$1, once per period



Deposit \$1/5, five times per period



Deposit \$1/30, thirty times per period



In this case, even though we are always looking at our balance over the same time interval, the larger m is, the more our balance function looks like a continuous line.

In the limit as $m \to \infty$, it makes sense to consider your balance to be given by $M(t) = 1 + t$. More generally, $M(t) = P + Dt$ if we had opened the account with P dollars, and deposited D dollars each period,

Notice that in this case, $M'(t) = D$. This equation is telling us that the rate of change of money in your bank account (in dollars per period) is always D. This means that your bank account is changing at a constant rate of D dollars per period, which is consistent with the situation.

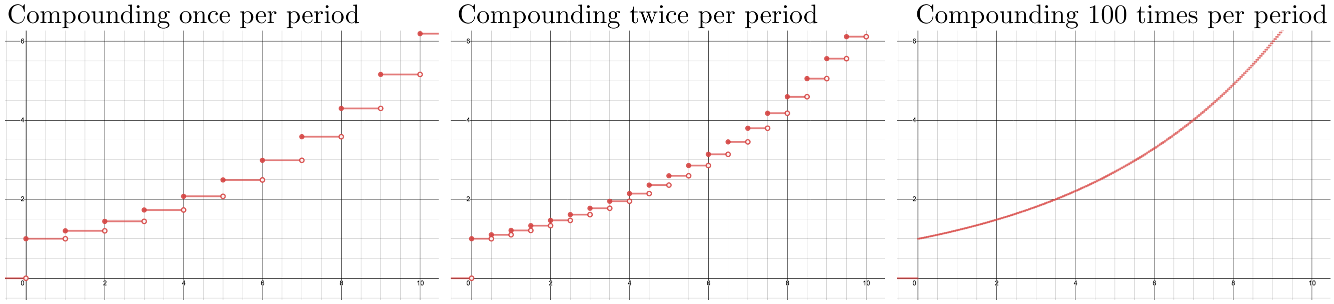
It is important to contrast with the case when m is finite (so we are not continuously depositing, but depositing m times per period). In that case, $M'(t)$ is equal to 0 everywhere, except where M has jumps, $M'(t)$ will be undefined. Even though we are still depositing a total of D dollars each period, the derivative is not able to give us any useful information in this case. All the derivative tells us in this case is that most of the time the balance is not changing (rate of change is equal 0 dollars per period), except at the jumps where it changes discontinuously and therefore not in a differentiable way.

2 Calculating continuously compounded interest

If interest is calculated once per period at interest rate r, then the balance at the end of the period will be $(1+r)$ times the balance at the beginning of the period. Thus, if P is the initial amount of money in the account, then $M(t) = P(1+r)^t$ describes the amount of money after any whole number of periods t. Notice that the formula doesn't make sense if t is not a whole number, since the balance does not change until the end of the period.

When we say interest is "compounded" m times per period, we mean that we split the period into m equal parts, and at the end of each sub-period, you are credited with with interest at rate r/m. Thus $M(t) = P\big((1+\frac{r}{m})^m\big)^t$. From the notes on the definition of e, we know that$(1+\frac{r}{m})^m \to e^r$ as $m \to \infty$, so if m is very large, or in the limiting case when the interest is compounded *continuously*, the balance will evolve according to $M(t) = Pe^{rt}$.

Compounding once per period | Compounding twice per period | Compounding 100 times per period



3 Exercises

1. Suppose you open a bank account with an initial deposit of \$100. If you do not earn any interest on the account, how much money will be in the account after 10.75 years in each of the following situations.

   (a) You deposit an additional \$20 every year, starting one year from when you opened the account.

   (b) You deposit an additional \$10 every 6 months, starting 6 months from when you opened the account.

   (c) You deposit money continuously at a rate of \$20 per year.

   (d) For each of (a),(b),(c), give a rough sketch of the graph of your balance as a function of time.

2. Suppose you open a bank account with an initial deposit of \$100. If you do not make any more deposits to the account, how much money will be in the account after 10 years if you earn interest at rate $r = 0.20$ in each of the following situations.

   (a) Interest is compounded once per year.

   (b) Interest is compounded five times per year.

   (c) Interest is compounded continuously.

   (d) For each of (a),(b),(c), give a rough sketch of the graph of your balance as a function of time.