Implicit Differentiation

The chain rule is a useful tool for quickly calculating many derivatives, but it also allows us to extend our understanding and skills with derivatives to more situations than a function of one variable.

1 The Chain Rule

If f and g are differentiable, then the chain rule says that the derivative of the composition $f \circ g$ can be written in terms of the individual derivatives f' and g' as follows:

$$\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'\big(g(x)\big) \cdot g'(x)$$

Note that in the special case when $f(x) = x^n$, the chain rule says that $\frac{d}{dx}\Big[\big(g(x)\big)^n\Big] = n\big(g(x)\big)^{n-1} \cdot g'(x)$

If we let y denote $g(x)$ and $\frac{dy}{dx}$ denote it's derivative $g'(x)$, this may be written as $\frac{d}{dx}\big[y^n\big] = ny^{n-1} \cdot \frac{dy}{dx}$

2 Implicit Differentiation

2.1 Taking the derivative on both sides of an equation

An important first step to understanding how implicit differentiation works is to realize that whenever we have an equation in one variable $L(x) = R(x)$, we can think of it as the statement that two functions $L(x)$ and $R(x)$ (whose formulas are the expressions on either side of the equation) are equal.

Once we know that the two functions are equal, it must be the case that their derivatives are also equal. This means we can take the derivative with respect to x on both sides of the equation and preserve the equality $L'(x) = R'(x)$.

There are some examples and exercises in the text and on WebAssign for section 2.3 that ask you to compute the derivative of a function in two different ways. For example, the function $L(x) = 2x^2(2x-5)$ may be expanded and written as $R(x) = 4x^3 - 10x^2$, so that

$$2x^2(2x-5) = 4x^3 - 10x^2$$

Since the functions on either side are equal, we can differentiate on both sides of the equation to obtain

$$\frac{d}{dx}\Big[\ 2x^2(2x-5)\ \Big] = \frac{d}{dx}\Big[\ 4x^3 - 10x^2\ \Big]$$

On the left, we can differentiate using the Product Rule, while on the right we can differentiate using only the Sum Rule and Constant Multiple Rules. Even though the expressions for the derivatives may look different, they must be the same (and can be shown to be the same through algebra).

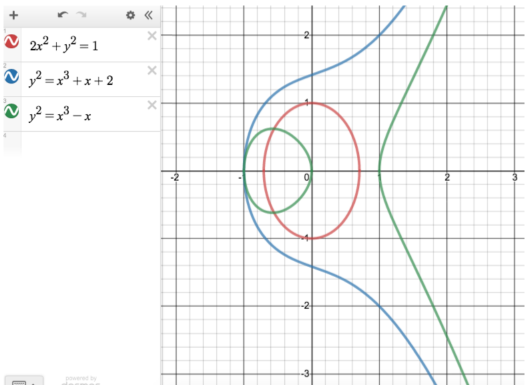
$$4x(2x-5) + 2x^2(2) = 12x^2 - 20x$$

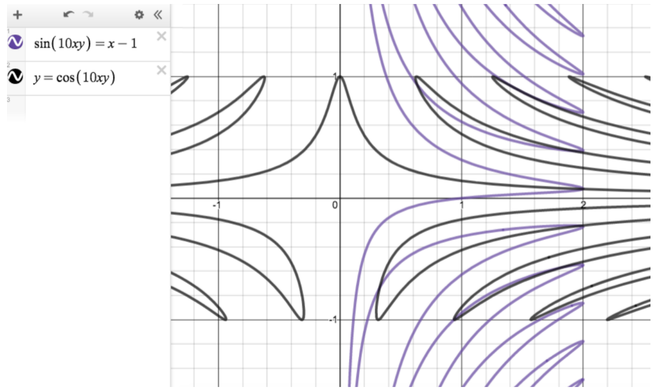
2.2 Equations in the variables x and y

Whenever you write down an equation in the **two variables** x and y, there are (possibly) some points in the xy-plane which make the equation true. In this way, we associate "curves" or "graphs" in the xy-plane with equations in the variables x and y.

For example, the unit circle centered at the origin in the xy-plane is the set of ordered pairs (x, y) which satisfy $x^2 + y^2 = 1$. More generally, the equation $(x - x_0)^2 + (y - y_0)^2 = R^2$ is the circle of radius R with center (x_0, y_0).

Here are some examples from Desmos of an ellipse in red, and some "elliptic curves" in blue and green, and some equations involving trig functions on the right:





Since each of these curves fails the vertical line test, none of the graphs are the graph of a function $y = g(x)$. Despite this fact, if we look locally near any point on any of the curves, the nearby part of the graph that passes through the point **does** look like the graph of a function, although if the graph has a vertical tangent line we cannot always express $y = g(x)$, but in this case we can express x as a function of y. Note that if the tangent line at a particular point is neither vertical nor horizontal, we can express either variable as a function of the other, $y = g(x)$ and $x = h(y)$, with $g^{-1} = h$.

In particular, even though none of these graphs are the graph of a function, all of these graphs have well-defined tangent lines at every point. Implicit differentiation is exactly the tool that let's us find tangent lines (derivatives), even when we have equations that are more complicated than the equation of a function $y = f(x)$.

3 Procedure to find $\frac{dy}{dx}$ for equations in x and y

0. Remember that y is implicitly a function of x and its derivative is denoted $\frac{dy}{dx}$. It may help at first to substitute $y = g(x)$, even though we will not have a formula for $g(x)$.

1. Differentiation both sides of the equation with respect to x.

   If you made the substitution $y = g(x)$, you only need to apply derivative rules correctly, and realize that your answer will depend on $g(x)$ and $g'(x)$, but these are just “explicit” formulas for y and $\frac{dy}{dx}$.

2. Use algebra to solve the resulting equation for $\frac{dy}{dx}$ or $g'(x)$, depending on your choice above.

   In any case, you will have a formula for the derivative that (probably) depends on both x and $y = g(x)$. Thus, in order to determine the slope of a tangent lines at a point on the curve, you need to know both of the (x, y) coordinates of the point.

4 Exercises

1. (a) For the ellipse $2x^2 + y^2 = 1$, find the implicit derivative $\frac{dy}{dx}$. Use your answer to determine where the graph has vertical and horizontal tangent lines.
   (b) Find the (x, y) coordinates of all points on the ellipse with $y = \frac{1}{2}$.
   (c) Use you asnwer from (a) to find the equations of the tangent line to the ellipse at each of your points in (b).

2. (a) For the curve $y = \cos(10xy)$, find the implicit derivative $\frac{dy}{dx}$. Use your answer to determine where the graph has vertical and horizontal tangent lines.
   (b) Find the (x, y) coordinates of all points on the curve in (a) with $y = \frac{1}{2}$.
   (c) Use you asnwer from (a) to find the equations of the tangent line to the curve at each of your points in (b).