Newton's Method for Finding Roots

Newton's Method is a powerful tool for numerically approximating solutions to many equations, and is the starting point for many more sophisticated numerical methods (see Math 309: Numerical Analysis).

1 Root Finding

Recall that a **root** or **zero** of a function f is a number x which makes $f(x) = 0$. Any time you are trying to solve an equation in one variable, you are equivalently trying to find a root of an associated function. Many problems can be translated into a root finding problem, to which Newton's method can be applied.

For example, **"the golden ratio"** is defined to be the positive number ϕ which satisfies the equation $\frac{\phi}{1} = \frac{\phi+1}{\phi}$. After some algebra we can show that this is equivalent to saying that $\phi^2 = \phi + 1$, or $\phi^2 - \phi - 1 = 0$. In other words, ϕ is a root of the quadratic polynomial

$$f(x) = x^2 - x - 1$$

As another example, **"the cube root of 7"** is defined to be the number $\sqrt[3]{7}$ which satisfies the equation $\left(\sqrt[3]{7}\right)^3 = 7$. In other words, $\sqrt[3]{7}$ is a root of the cubic polynomial

$$f(x) = x^3 - 7$$

More generally, whenever we seek a number x which makes a certain equation true $L(x) = R(x)$, we are really asking for a root of the function $f(x) = L(x) - R(x)$, since $f(x) = 0$ if and only if $L(x) = R(x)$.

2 Root Finding Methods

Here we present two root finding methods. The Bisection Method requires that the function f is continuous, while Newton's Method additionally requires that it is differentiable.

2.1 The Bisection Method

The Bisection Method is based on a special case of the Intermediate Value Theorem, which says that if $f(a)$ and $f(b)$ have opposite sign (one is positive and the other negative), then f has a root in the interval (a, b).

In order to use the Bisection Method, you need an initial choice of a and b that makes f change sign, and for f to be continuous on the interval $[a, b]$. Call these initial values a_0 and b_0.

At each stage, you will use your current interval $[a_n, b_n]$ to produce a new interval $[a_{n+1}, b_{n+1}]$, which is half the length of the previous interval. The method works by calculating the midpoint of the interval, $c_n = \frac{a_n+b_n}{2}$, and evaluating the function at the midpoint $f(c_n)$.

How you proceed depends on whether $f(c_n)$ is zero, positive, or negative. If it is zero, you have found the root of f! Otherwise, if $f(c_n)$ has sign opposite the sign of $f(a_n)$, we know by IVT that the root is in the interval $[a_n, c_n]$. In the only other case, $f(c_n)$ must have sign opposite the sign of $f(b_n)$, in which case we again know by IVT that the root is in the interval $[c_n, b_n]$. In either case, we have found a new smaller interval which is guaranteed to contain the root of f. We call this new interval $[a_{n+1}, b_{n+1}]$ and repeat the process.

2.2 Newton's Method

If f is differentiable, we can do better than the Bisection Method, by using tangent lines to produce approximations whose error decrease much more rapidly per iteration than the halving of the error that occurs with the Bisection Method.

In order to use Newton's method, f needs to be differentiable, and you need to choose an "initial guess" for the root, denoted x_0, which is ideally nearby the root.

At each stage of the method, you construct the tangent line to f at $x = x_n$, and use the root of the tangent line as your next guess. Since the equation of the tangent line is given by

$$y = f(x_n) + f'(x_n)(x - x_n)$$

when we set $y = 0$ and solve for x, we obtain the formula which captures how the algorithm proceeds:

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

We call this new x-value x_{n+1} and repeat the process.

3 Comparison of the two methods

It is important to note that for both methods, you are never guaranteed to produce the exact solution, but rather to generate a sequence of estimates, which get closer and closer to the exact answer.

The Bisection Method is guaranteed (by IVT) to produce a sequence of intervals which "converge to the root" (since the distance between a_n and b_n goes to 0 in the limit as $n \to \infty$, and the root is in every interval $[a_n, b_n]$), but the convergence happens relatively slowly. If your interval has length 0.01, after 10 more iterations, your interval will have length $0.01/2^{10} \approx 0.00001$.

Newton's Method is not guaranteed to converge, but when it does it has what is known as "quadratic convergence", which means that once you are close enough to the root, your subsequent error will be proportional to the square of your current error. So if you current error is 0.01, on the next iteration your error will be on the order of $(0.01)^2 = 0.0001$, and after only one more iteration, the error will be on the order $(0.0001)^2 = 0.00000001$.

Although Newton's Method converges much faster, it is not guaranteed to work until your initial guess is near enough to the root. But for some functions, like polynomials, Newton's Method is guaranteed to converge to a root no matter what your initial guess was (as long as you don't encounter a point which makes $f'(x_n) = 0$), though this is not always the case.

It is easy to tell that Newton's Method is converging in practice, if you have access to the decimal approximations of the x_n. If it is converging, the numbers will start agreeing to more and more decimal places, guaranteeing that the approximation is accurate to at least that many decimal places.

4 Exercises

1. For each of the examples above $f(x) = x^2 - x - 1$ and $f(x) = x^3 - 7$, use two iterations of each method, starting with the interval $[1, 2]$ for the Bisection Method, and the initial guess $x_0 = 2$ for Newton's Method. (Try to do it all by hand, without using a calculator.)

2. Use the quadratic formula and a calculator to determine the true value of the golden ratio and $\sqrt[3]{7}$ to 8 decimal places. Compare with your approximations from 1.

3. For each of the examples in 1, sketch a graph of f near its root, and label the relevant intervals used for the first iteration of the Bisection Method, as well as the relevant point and tangent line used for the first iteration of Newton's Method.

4. Approximating the value of any square root $\sqrt{d}$ amounts to approximating the root of the function $f(x) = x^2 - d$. Use algebra to show that in this case, the formula for the Newton's Method iteration for updating your guess may be expressed as follows, in terms of your previous guess x_n

$$x_{n+1} = \frac{x_n + \frac{d}{x_n}}{2}$$

5. Find a similar simplified formula to the one in 4. for approximating the cube root $\sqrt[3]{d}$, ie. the root of the cubic $f(x) = x^3 - d$.