Math 131 - Precalculus Review Problems

These problems review some important precalculus concepts relating to functions and their graphs, especially linear, quadratic, and piecewise functions, function composition, inverses, domains, and interpretation of your answers. For most parts, you final answer should be written in complete sentences. Try to complete the problems without using any resources besides a simple (non-graphing) calculator for arithmetic. If you get stuck, try graphing the relevant functions or points on Desmos or a graphing calculator.

1 Temperatures

If x represents temperature in degrees Celsius, and y represents temperature in degrees Fahrenheit, then we can find the conversion formula between the units and represent it as a function $y = f(x)$ by knowing only the following facts:

- The function f is linear, ie. the graph of $y = f(x)$ is a line.
- $f(0) = 32$ and $f(100) = 212$ (the freezing and boiling points for water).

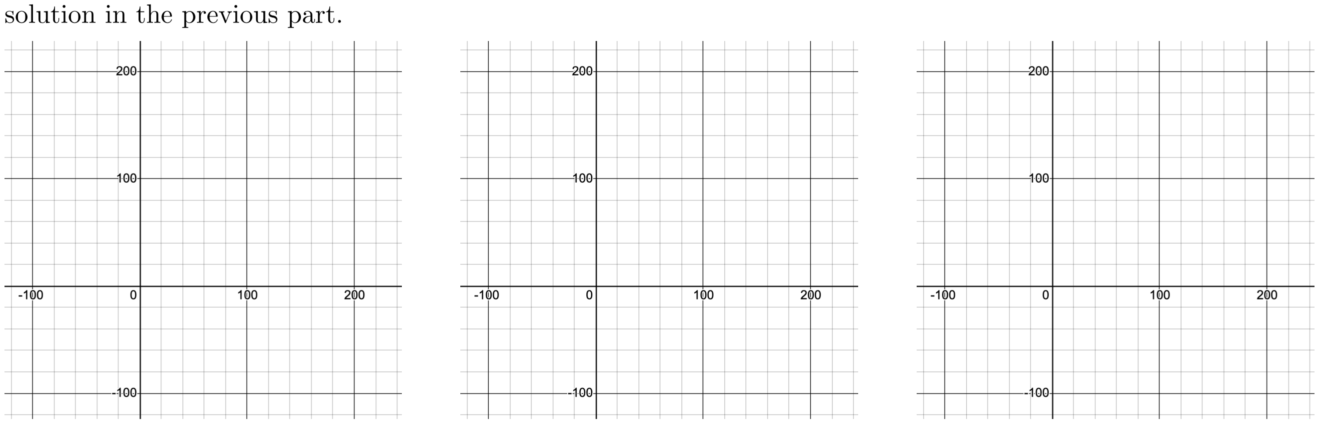
1. Find the formula for $f(x)$. Sketch its graph on the left below. Make sure to label the axes on all graphs. What are the slope and y-intercept of this line? What do these numbers represent in the context of temperatures?

2. Find the formula for $f^{-1}(x)$. What does x represent in this formula? Sketch its graph in the middle below. What are the slope and y-intercept of this line? What do these numbers represent in the context of temperatures?

3. Use algebra to find the (x, y) coordinates for the point of intersection between the graphs of f and f^{-1}. What is the significance of this point?

4. Using the appropriate function above, find the temperature in Celsius of a normal human body, given that it is 98.6 in degrees F.

5. On the right, graph the horizontal line $y = 98.6$ along with one of the functions above to demonstrate your solution in the previous part.



6. Now suppose the function $T(x) = 70 + 5x$ represents the temperature in degrees Farhenheit of the pavement outside, x hours after 6 AM this morning.

   (a) At least one of the functions below makes sense in terms of the situation. Determine which one(s), and find formulas for them, explaining what the formula and its variables represent. For the others, explain why they don't make sense. Recall that the composition $g \circ h$ of two functions g and h is a new function defined by $(g \circ h)(x) = g(h(x))$.

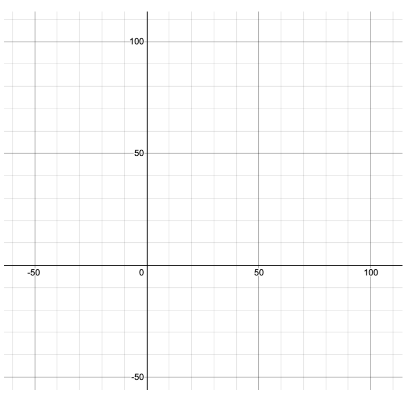
   $$f \circ T \qquad f^{-1} \circ T \qquad T \circ f \qquad T \circ f^{-1}$$

   (b) Find the temperature of the pavement at 6 PM in Fahrenheit, and use your answer to the previous part to find it in Celsius.

7. If the wind is blowing at 20 miles per hour, the "wind chill temperature" y in degrees Fahrenheit can be calculated from the current temperature x in degrees Fahrenheit via the functional relationship $y = W(x)$, where $W(x) = 1.3x - 22$.

   (a) If its -10 degrees F and the wind speed is 20 mph, what temperature does is feel like?

   (b) Graph $y = W(x)$ along with the line $y = x$. Compare the graphs and determine if the function W makes sense for all values of x. Hint: What does it mean if the graphs cross?



   (c) By composing W with f and/or f^{-1}, find a function that calculates the 20 mph wind chill temperature y from the current temperature x, with all temperatures in Celsius.

2 Motion

Later in the course, we will see that parabolas model the motion of moving objects due to the force of gravity (the trajectory of a falling apple or a planet for example).

Consider the function $s(t) = 6 + 5t - t^2$.

If t represents time in seconds, $y = s(t)$ could represent the height in feet above the ground of an apple thrown vertically upwards at time $t = 0$.

1. Find the roots of the quadratic $s(t)$. What do these numbers represent? Complete the square to represent $s(t) = k - (t - h)^2$ for appropriate constants h and k.

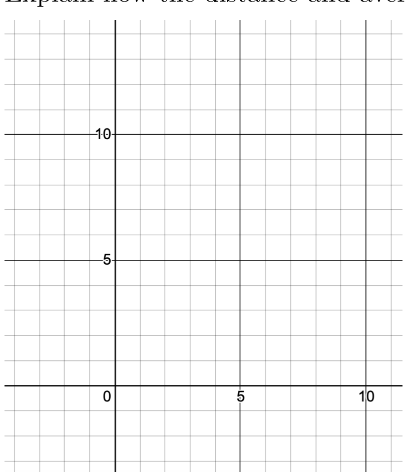
2. Find the time at which the apple reaches its maximum height. (Recall that the vertex of the parabola $y = ax^2 + bx + c$ is located at $x = -b/2a$, which is obtained from the completed square.)

3. What is the maximum height reached by the apple?

4. How much distance does the apple cover over the last second before it hits the ground? What is its average velocity over that same time interval. (Recall average velocity is the change in position divided by the change in time, "distance = rate × time".)

5. Given the situation, give a sensible domain for the function s. Explain why you chose this domain. State your domain using both interval notation and inequalities.

6. Sketch the graph of $y = s(t)$ on the domain you selected, and identify on the graph all points relevant to the previous parts.
7. Outside of the domain you chose, sketch a plausible graph of s, given the situation.
8. Explain how the distance and average velocity in the previous part can be represented on the graph.



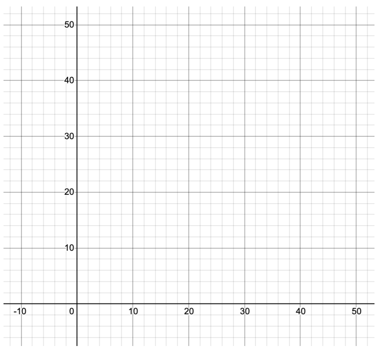
3 Cost functions

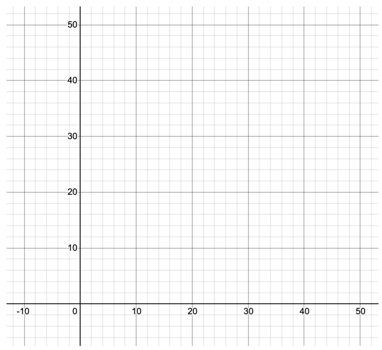
Suppose a gold distributor sells gold at \$1,500 USD per ounce for the first 20 ounces, and \$1,000 USD per ounce for each additional ounce beyond 20 ounces.

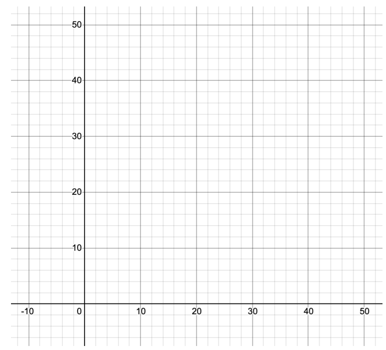
1. If $y = C(x)$ is the cost in thousands USD to purchase x ounces of gold from this distributor, find a piecewise formula for C.

$$C(x) = \begin{cases} \\ \\ \end{cases}$$

2. Sketch the graph of $y = C(x)$ on the left.







3. What is the average cost per ounce if you purchase 10 ounces? 20 ounces? 30 ounces?

4. If you start out with \$40,000 USD, let $B(x)$ represent the amount of money (in thousands of USD) you have left after purchasing x ounces of gold from this distributor. Find a piecewise formula for B and sketch the graph of $y = B(x)$ in the middle above. Explain how the graph of B can be obtained from the graph of C using "graph transformations" (vertical/horizontal shifts/strecthes).

$$B(x) = \begin{cases} \\ \\ \end{cases}$$

5. Choose a reasonable domain for each of the functions B and C. Explain your choices.

6. A different distributor charges \$1,600 USD per ounce if you buy 25 ounces or less, and charges \$1,200 USD per ounce if you buy more than 25 ounces. If this distributors cost function is $y = D(x)$, find a piecewise formula for D. Sketch the graph of D on the right.

$$D(x) = \begin{cases} \\ \\ \end{cases}$$

7. Which distributor should you buy from if you want to buy 20 ounces? 30 ounces?

8. Determine if each of the functions C and D is invertible. If it is, find a formula for the inverse, and explain what it represents. If it isn't, explain why.

Math 131 - Precalculus Skills Review

Many of the following questions are routine exercises from high school algebra and precalculus classes. You should be able to show your work and explain why it is correct, without needing to use a calculator for anything beyond basic arithmetic. There are videos in the Ch 0 playlist on the YouTube page that review many of the skills and ideas needed for these problems.

1. Simplify the expression $\sqrt[4]{48a^3b^8c^4}$. For which choices of a, b, c is this expression defined?

2. $a^6 - b^6$ equals (a) $(a^3 - b^3)^2$ (b) $(a^2 - b^2)^3$ (c) $(a - b)(a^5 + b^5)$ (d) none of these

   Note: the above can be factored with using the "difference of squares" formula. What is the resulting expression? Check your answer by using FOIL ie. distributing.

3. $\left(\frac{-8c^3}{c^{-6}}\right)^{2/3}$ equals (a) $4c^6$ (b) $64c^{-2}$ (c) $2c^{27/2}$ (d) none of these

4. Simplify the following expression by finding a common denominator: $\frac{\frac{1}{a+b} - \frac{1}{a}}{b}$. What happens in the resulting expression when $a = 1$ and $b = 0$? What about in the original expression?

5. Simplify the following expression by rationalizing the denominator: $\frac{\sqrt{t}+5}{\sqrt{t}-5}$. What happens in the resulting expression when $t = 25$? What about in the original expression?

6. Factor the following polynomial completely: $2x^3 - 4x^2 - 30x$.

   Use the following information for questions 8-10:

   $f(x) = \frac{x+1}{x-1}$, $g(x) = \sqrt{x-1}$, $h(x) = \begin{cases} 3x^2, & \text{if } x > 2 \\ x - 5, & \text{if } x \leq 2 \end{cases}$

7. Sketch the graph of each function, and note its domain. Compute $h(g(6))$ and $f(f(-2))$. Also, find the equation of the line which intersects the graph of f at $x = 2$ and $x = 3$. What is its slope?

8. Find a formula for the composition of functions, $(f \circ f)(x) = f(f(x))$. Use it to verify your computation of $f(f(-2))$ from 8.

9. Find a formula for the inverse function, $f^{-1}(x)$. What is it's domain and range?

10. Solve the quadratic equation $x^2 - 6x - 16 = 0$ by three different methods: (1) factoring, (2) using the quadratic formula, (3) completing the square.

11. Solve the system of equations $\left\{ \begin{array}{r} x^2 + 2y = 30 \\ 2x + y = 15 \end{array} \right\}$ for x and y. What do these solutions correspond to gemoetrically? (Hint: solve each equation for y)

12. Solve $|3x + 3| - 3 = 3$ for x. Sketch the graph of $f(x) = |3x + 3| - 3$ and explain what the solutions represent geometrically?

    Recall that the absolute value of x is calculated via the piecewise formaula $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

13. Find an equation of the line passing through the points $\left(-\frac{2}{3}, 0\right)$ and $\left(0, \frac{7}{2}\right)$. What is its slope? Find the equation of the line parallel to this one which passes through the point $(1, 2)$.

14. Which of the following lines are parallel to the line $3x - 8y = 4$? Circle all that apply:

    (a) $3x - 8y = 0$ (b) $8x - 3y = 2$ (c) $y = \frac{3}{8}x + 1$ (d) $-3x + 8y = 5$ (e) $y = -\frac{1}{2} - \frac{3}{8}x$

15. Sketch the graphs and note the differences in the graphs among the following functions. Note: once you have the graph of f, you should be able to use it to sketch graphs of the rest.

    $f(x) = \cos(x)$, $g(x) = \cos(2x)$, $h(x) = 2\cos(x)$ $k(x) = \dfrac{1}{\cos(x)}$