Slope Fields

1 Differential Equations

In this class, we will only focus on differential equations which can be put in the following form:

$$\frac{dy}{dt} = f(t, y)$$

$f(t, y)$ on the right is a formula that depends on the independent variable t and the unknown function $y = y(t)$. A more explicit way of writing what the differential equation is saying is: for all values of the independent variable t, the two numbers on both sides of the following equation are equal:

$$y'(t) = f(t, y(t))$$

Notice that the expression on the left and right are both functions of t. It should be clear that $y'(t)$ on the left is a function of t, since this is the standard notation for the derivative of $y(t)$. It is less clear that the expression on the right is also a function of t, since it seems to have two variables t and y. Really, the equation is telling us how to combine the numbers t and $y(t)$ in order to get $y'(t)$.

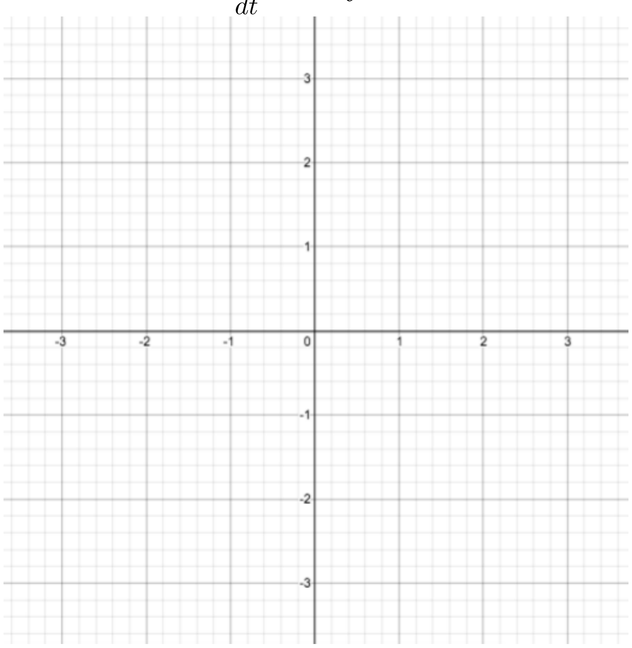
2 Slope Fields

Since the differential equation is telling us how at each point, the derivative of the solution depends on t and $y = y(t)$, if we know that the solution passes through a point $(t, y) = (t_*, y_*)$, so that $y(t_*) = y_*$, then the slope of the tangent line to y at t_* must be given by $y'(t_*) = f(t_*, y_*)$, ie. by plugging the coordinates on the point into the right hand side of the differential equation.

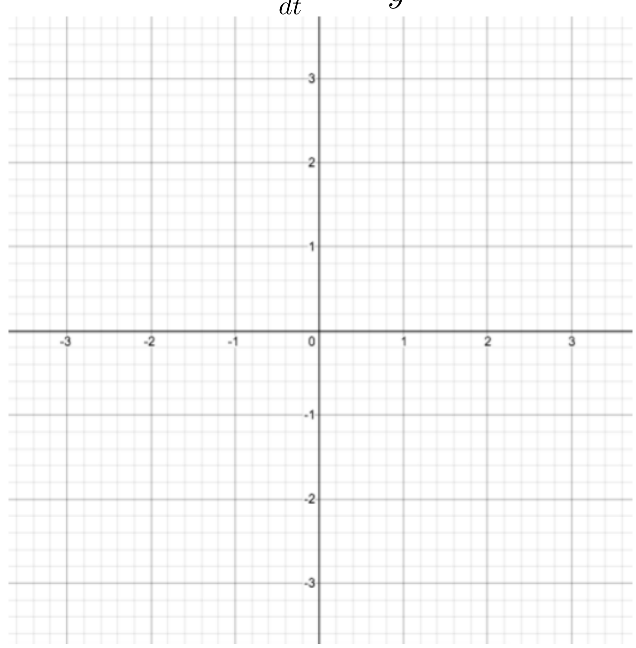
Since it is not difficult to evaluate the formula $f(t, y)$ at as many points as we like, we can determine ahead of time what the slopes would be at as many points as we like. By filling in a graph with a "minitangent" (a line segment whose slope is given by the differential equation) at each point on a predetermined grid, we can already get a good sense of how solutions will behave since solutions must "follow the slope field", ie. at any point $(t, y(t))$, a solution $y(t)$ must pass through that point with a slope that matches the slope field.

On each of the blank graphs below, use the differential equation to determine and draw the slope at each point (t, y) with $t = -3, -2, -1, 0, 1, 2$ and $y = -3, -2, -1, 0, 1, 2$.

$$\frac{dy}{dt} = -t$$



$$\frac{dy}{dt} = -y$$



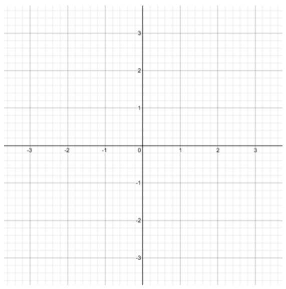
3 Properties of slope fields

3.1 Special Case: $\frac{dy}{dt} = f(t)$

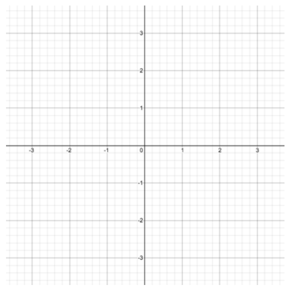
If the function on the right hand side only depends on t, then the differential equation is telling us that the solution $y(t)$ is any function whose derivative is $f(t)$. In Calc 2 you will learn that y is called an "antiderivative" of f, and you will spend Ch 5 of the text learning advanced techniques for solving this type of problem. Until then, it is still possible to use your understanding of derivatives to work backwards and solve lots of simple problems of this type.

Exercise: Sketch the slope field for each of the following differential equations. Use your knowledge of derivatives to work backwards and find a formula for the solution $y(t)$. If you are stuck, use the slope field to help you guess the right solution.

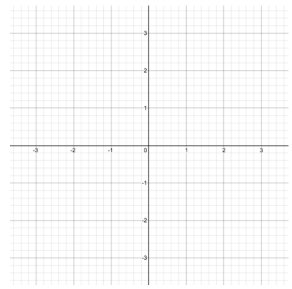
$$\frac{dy}{dt} = 2$$



$$\frac{dy}{dt} = 2t$$



$$\frac{dy}{dt} = 2t^2$$



Notice that whenever the differential equation depends on t only, the slope field does not change when you move vertically on the plane. So once you know how the slope field looks on the line $y = 0$ (or any other horizontal line), you can shift it to fill out the rest of the plane.

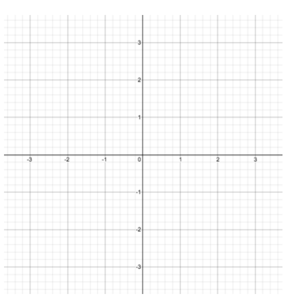
This behavior is exactly related to **Theorem 3.7** in the text, which says that if two functions have the same derivative, then they differ by a constant. If y_1 and y_2 are both solutions to the differential equation, then $y_1'(t) = f(t)$ and $y_2'(t) = f(t)$. Since y_1 and y_2 have the same derivative, it must be that $y_1 = y_2 + C$ for some constant C, so all of the solutions are vertical shifts of each other.

3.2 Special Case: $\frac{dy}{dt} = f(y)$

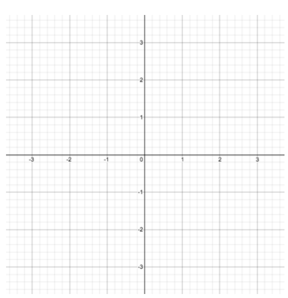
If the function on the right hand side only depends on y, then the differential equation is telling us that the derivative y' can be determined by only knowing the y value, and does not depend on t. These are much more interesting equations than the ones above, because they give a relationship between y and y', instead of just specifying that y' needs to be equal to some particular $f(t)$.

Exercise: Sketch the slope field for each of the following differential equations. Try to use the slope field to guess (and check) at least one solution for each. Since the first one is an exponential decay model, write down a formula for all the different solutions.

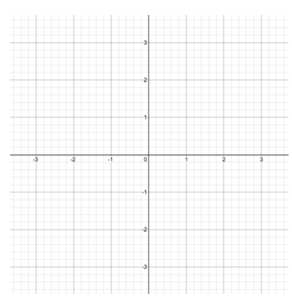
$$\frac{dy}{dt} = -2y$$



$$\frac{dy}{dt} = 1 - 2y$$



$$\frac{dy}{dt} = 2y - 1$$



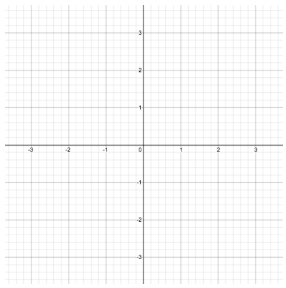
Notice that whenever the differential equation depends on y only, the slope field does not change when you move horizontally on the plane. So once you know how the slope field looks on the line $x = 0$ (or any other vertical line), you can shift it to fill out the rest of the plane.

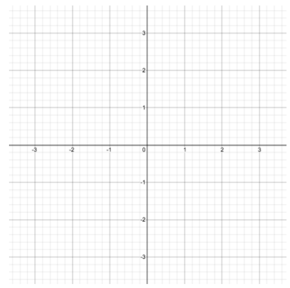
3.3 General Case: $\frac{dy}{dt} = f(t, y)$

In general, we can have any expression in the variables t and y on the right hand side of the differential equation. In this case, we shouldn't expect any symmetry when we shift vertically or horizontally, but we can still use the slope field to quickly get a sense of how the solutions behave.

Exercise: Sketch the slope field for each of the following differential equations. Try to use the slope field to guess (and check) at least one solution for each.

$$\frac{dy}{dt} = ty \qquad \frac{dy}{dt} = t - y$$



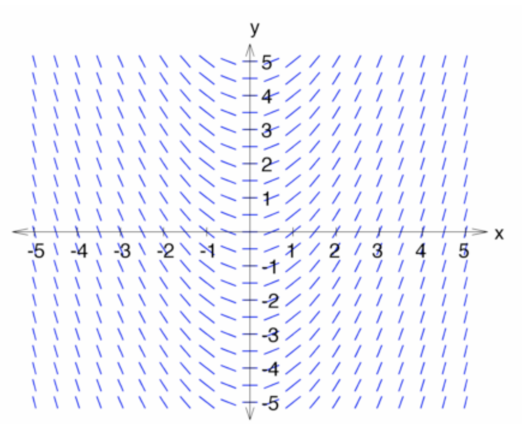


1. Match each of the following slope fields to one of the differential equations below.

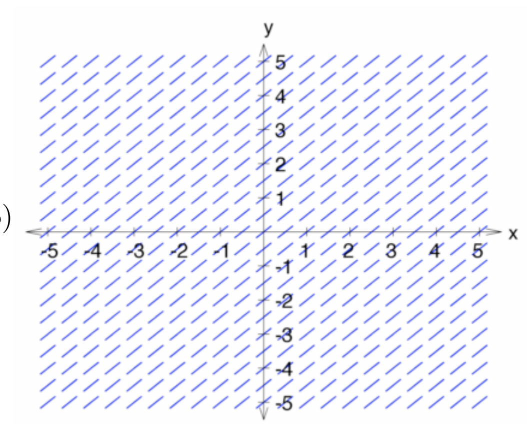
(i) $\dfrac{dy}{dx} = x^2$ (ii) $\dfrac{dy}{dx} = -x$ (iii) $\dfrac{dy}{dx} = 1$

(iv) $\dfrac{dy}{dx} = -x^2$ (v) $\dfrac{dy}{dx} = -1$ (vi) $\dfrac{dy}{dx} = x$

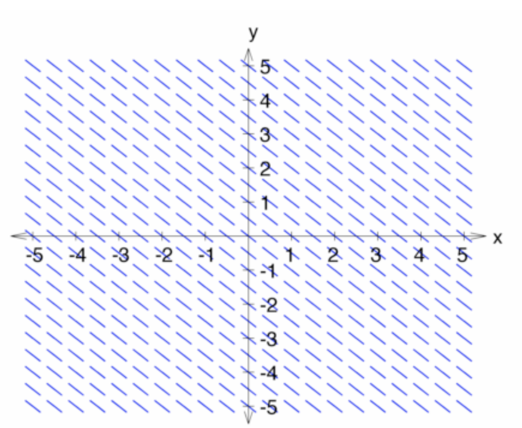
(a)



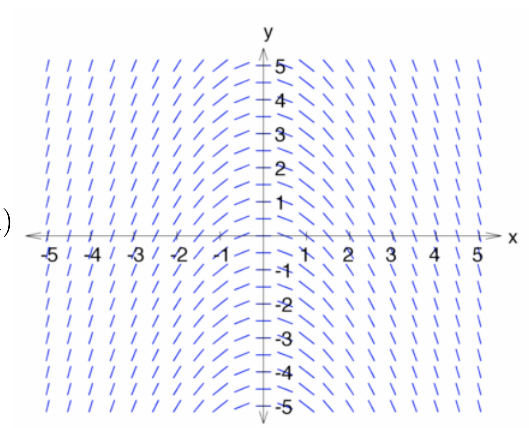
(b)



(c)



(d)



2. On each of the slope fields above, sketch the solution which satisfies $y(0) = 0$.

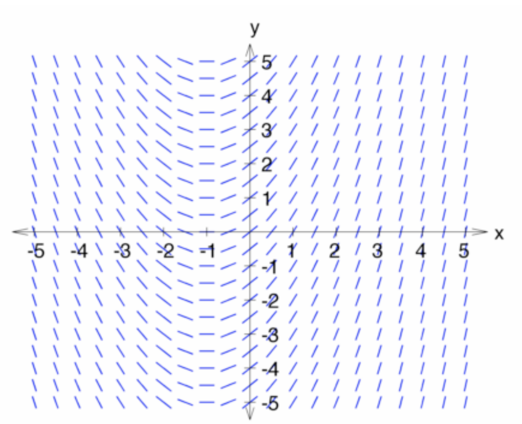
3. On each of the slope fields above, sketch the solution which satisfies $y(0) = 2$.

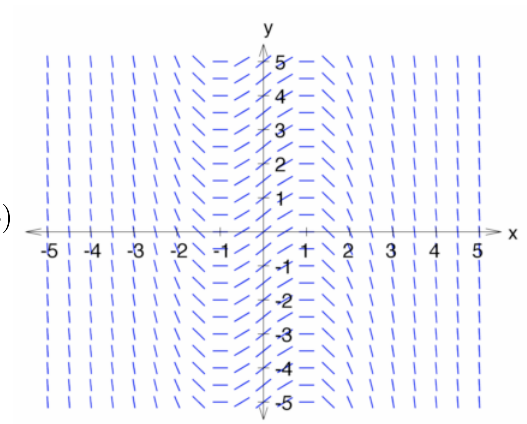
4. On each of the slope fields above, sketch the solution which satisfies $y(0) = -2$.

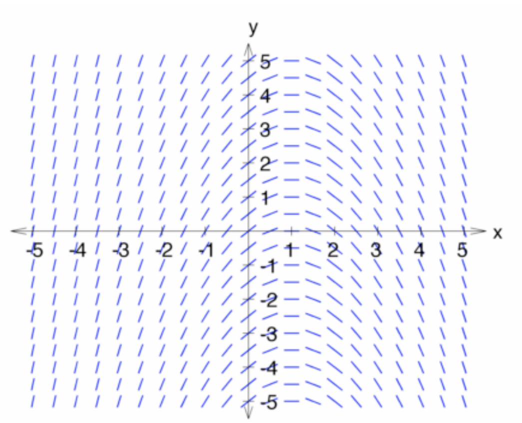
5. Match each of the following slope fields to one of the differential equations below.

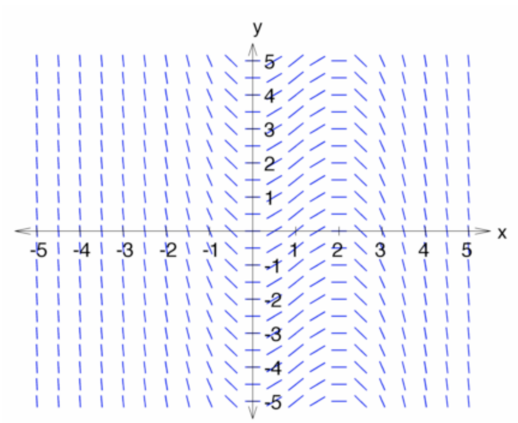
(i) $\dfrac{dy}{dx} = x^2 + x$ (ii) $\dfrac{dy}{dx} = 1 - x$ (iii) $\dfrac{dy}{dx} = 2x - x^2$

(iv) $\dfrac{dy}{dx} = 1 - x^2$ (v) $\dfrac{dy}{dx} = x - x^2$ (vi) $\dfrac{dy}{dx} = x + 1$

(a)


(b)


(c)


(d)


6. On each of the slope fields above, sketch the solution which satisfies $y(0) = 0$.

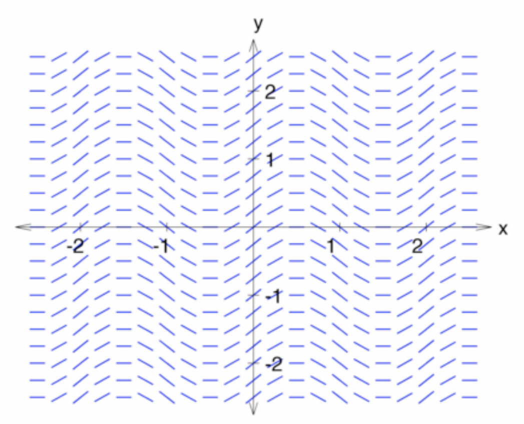
7. On each of the slope fields above, sketch the solution which satisfies $y(0) = 2$.

8. On each of the slope fields above, sketch the solution which satisfies $y(0) = -2$.

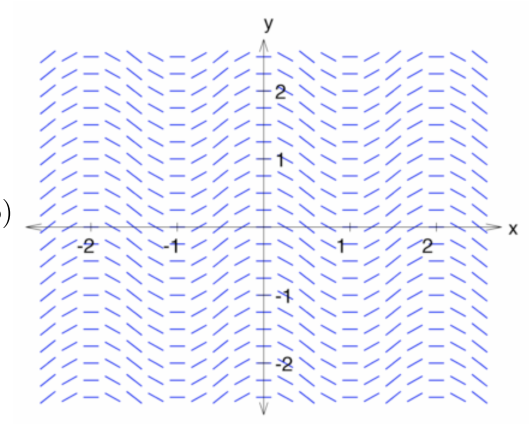
9. Match each of the following slope fields to one of the differential equations below.

(i) $\dfrac{dy}{dx} = \sin(\pi x)$ (ii) $\dfrac{dy}{dx} = \cos(\pi x)$ (iii) $\dfrac{dy}{dx} = -\sin(\pi x)$ (vi) $\dfrac{dy}{dx} = -\cos(\pi x)$

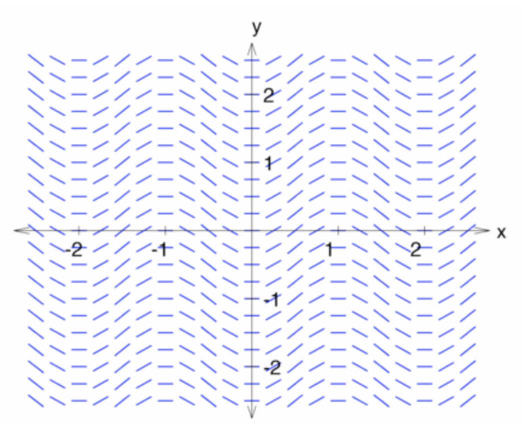
(a)



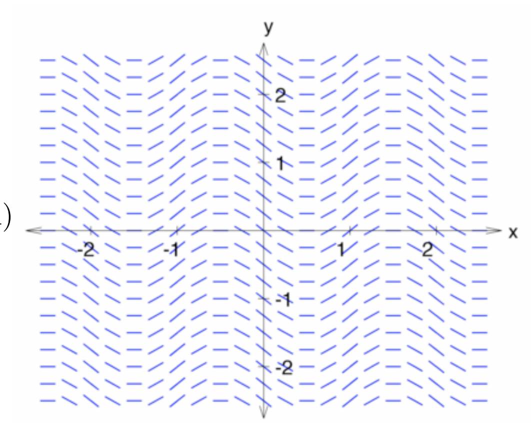
(b)



(c)



(d)



10. On each of the slope fields above, sketch the solution which satisfies $y(0) = 0$.

11. On each of the slope fields above, sketch the solution which satisfies $y(0) = 1$.

12. On each of the slope fields above, sketch the solution which satisfies $y(0) = -1$.